LECTURES ABOUT (ADVANCED) STATISTICAL PHYSICS

T.S.Biró, *MTA Wigner Research Centre for Physics, Budapest*

Lectures given at: University of Johannesburg, South-Africa,

November 26 – November 29, 2012.

1. **Ancient Thermodynamics (… - 1870)**
2. **The Rise of Statistical Physics (1890 – 1920)**
3. **Modern (postwar) Problems (1940 – 1980)**
4. **Corrections (1950 – 2005)**
5. **Generalizations (1960 – 2010)**
6. **High Energy Physics (1950 – 2010)**

LECTURE TWO ABOUT (ADVANCED) STATISTICAL PHYSICS

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Kinetic theory

- **Sum of random forces: noise (Gaussian)**
- **Brownian motion**

TSB, CG, PRL 79, 3138, 1997

- **Langevin and Fokker-Planck equations**
- **Fluctuation-dissipation theorem**
- **Boltzmann equation**
- **Entropy → equilibrium theory**

TSB, PG, hep-ph/0503204

General Langevin problem

$\dot{p} = F(p, z)$

Wang + Uehlenbeck:

use R(p) test function!

Many p(t) evolutions from p(0): f(p,t) distribution

$$\int dp\, R(p)\, f(p, t+dt) = \int dp \left\langle R(p + dt\, F(p,z)) \right\rangle f(p,t)$$

average over noise $< F > = - G(p),$

$< F\, F > - < F > < F > = 2\, D(p) \,/\, dt$

General Langevin problem

Expansion til o(dt) gives:

$$\int dp\, R(p)\frac{\partial f}{\partial t} = \int dp\left[-R'(p)G(p) + R''(p)D(p)\right] f(p,t)$$

Fokker-Planck equation after partial integration:

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial p}\left(G(p)\, f\right) + \frac{\partial^2}{\partial p^2}\left(D(p)\, f\right)$$

Particular Langevin problem

TSB, GGy ,AJ, GP, JPG31, 759, 2005

$$\dot{p} = z - G(E)\frac{\partial E}{\partial p}$$

$$< z(t) > \; = \; 0$$

$$< z(t)z(t') > \; = \; 2\, D(E)\, \delta(t-t')$$

In the Fokker – Planck equation:

$$D(p) = D(E)$$

$$G(p) = -G(E)\frac{\partial E}{\partial p}$$

Stationary distribution:

$$f(p) = \frac{A}{D(E)} \exp\left(-\int G(E)\frac{dE}{D(E)}\right) = A \exp\left(-\int \frac{dE}{T(E)}\right)$$

Inverse logarithmic slope → *temperature*

$$\frac{1}{T(E)} = -\frac{d}{dE} \ln f(E)$$

$$T(E) = \frac{D(E)}{G(E) + D'(E)}$$

$T_{Einstein} = D(E) / G(E)$

$T_{Gibbs} = D(0) / G(0)$

General inverse slope

Stationary distribution:

$$f(p) = A \exp\left(-\int \frac{dE}{T(E)}\right)$$

1) Gibbs: $T(E) = T \rightarrow \exp(-E/T)$

2) Tsallis: $T(E) = T/q + (1-1/q)\,E \rightarrow$

$$\left(1 + (q-1)\,E/T\right)^{-q/(q-1)}$$

$T(\mathrm{T}) = \mathrm{T}$: a fixed point of the sliding slope

Fluctuation ⟷ Dissipation *theorem*

$$D_{ij}(E) = T(E)\left(G_{ij}(E) + D'_{ij}(E)\right)$$

(*Hamiltonian eom does not change energy E!*)

$$\dot{p}_i = -G_{ij}\nabla_j E + z_i$$

$$D_{ij}(E) = \frac{1}{f(E)}\int_E^{\infty} G_{ij}(x)\, f(x)\, dx$$

with f(E) stationary distribution

Fluctuation ⟺ Dissipation *theorem*

particular cases (for constant G):

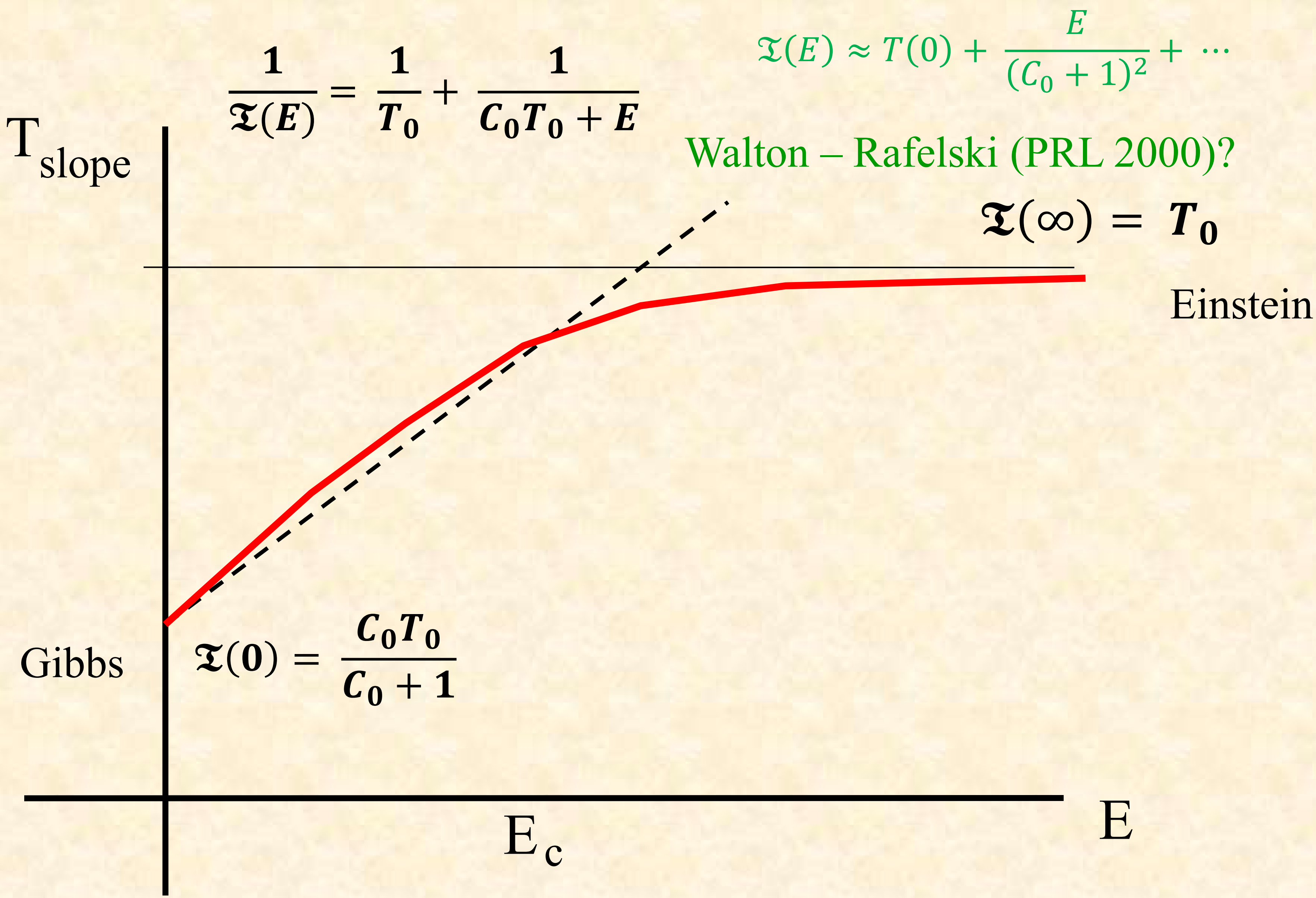
Gibbs: $$D_{ij} = T\ G_{ij}$$

Tsallis: $$D_{ij}\ (E) = \Big(T + (q-1)\ E \Big)\ G_{ij}$$

Field theory calculation

- **polynomial interaction, one field integrated out**
- **Imaginary part of self-energy → noise**
- **Effective Langevin eq. for soft field**
- **Fluctuation-dissipation:** $D_{ij} = T_0\, G_{ij}$ **(*constant Einstein temperature)***
- **G is linear in the energy:** $G_{ij} = \gamma_{ij}(1 + E/T_0 C_0)$
- **f(E) Gaussian**

$$\frac{1}{\mathfrak{T}(E)} = \frac{1}{T_0} + \frac{1}{C_0 T_0 + E}$$


$$\frac{1}{\mathfrak{T}(E)} = \frac{1}{T_0} + \frac{1}{C_0 T_0 + E}$$
$$\mathfrak{T}(E) \approx T(0) + \frac{E}{(C_0+1)^2} + \cdots$$
Walton – Rafelski (PRL 2000)?
$$\mathfrak{T}(\infty) = T_0$$
T_{slope}
Einstein
Gibbs
$$\mathfrak{T}(0) = \frac{C_0 T_0}{C_0 + 1}$$
E_c
E

Additive and multiplicative noise

Equivalent descriptions:

TSB, AJ, PRL 94, 132302, 2005

1. Langevin

$$\dot{p} = \zeta - \gamma p$$

$\langle \gamma \rangle = G$ $\quad \langle \zeta \rangle = F$

$\langle \gamma\gamma \rangle_c \sim 2C$ $\quad \langle \gamma\zeta \rangle_c \sim 2B$ $\quad \langle \zeta\zeta \rangle_c \sim 2D$

2. Fokker Planck

$$\frac{\partial f}{\partial t} = \frac{\partial^2}{\partial p^2}(K_2 f) - \frac{\partial}{\partial p}(K_1 f)$$

$K_1 = F - Gp$

$K_2 = D - 2Bp + Cp^2$

Exact stationary distribution:

$$f = f_0 \, (D/K_2)^v \; \exp\left(-\frac{\alpha}{\theta} \, \operatorname{atan}\left(\frac{\theta \, p}{D - Bp}\right)\right)$$

with

$v = 1 + G/2C$ *power*

$\alpha = GB/C - F$ *exponent*

$\theta^2 = DC - B^2$ *(small or large) parameter*

For F = 0 characteristic scale: $p_c^2 = D/C$.

Exact stationary distribution for F = 0, B = 0:

$$f = f_0 \left(1 + \frac{C\,p^2}{D} \right)^{-(1+G/2C)}$$

With E = $p^2 / 2m$ this is a Tsallis distribution!

$$f = f_0 \left(1 + (q-1)\frac{E}{T} \right)^{\frac{q}{1-q}}$$

- Tsallis index: $q = 1 + 2C / G$
- Temperature: $T = D / mG$

Limits of the Tsallis distribution:

$p \ll p_c$: Gauss

$$f \sim \exp(- Gp^2 / 2D)$$

$p \gg p_c$: Power-law

$$f \sim (p / p_c)^{-2v}$$

Energy distribution limits:

$E \ll E_c$: $f \sim \exp(-E/T)$

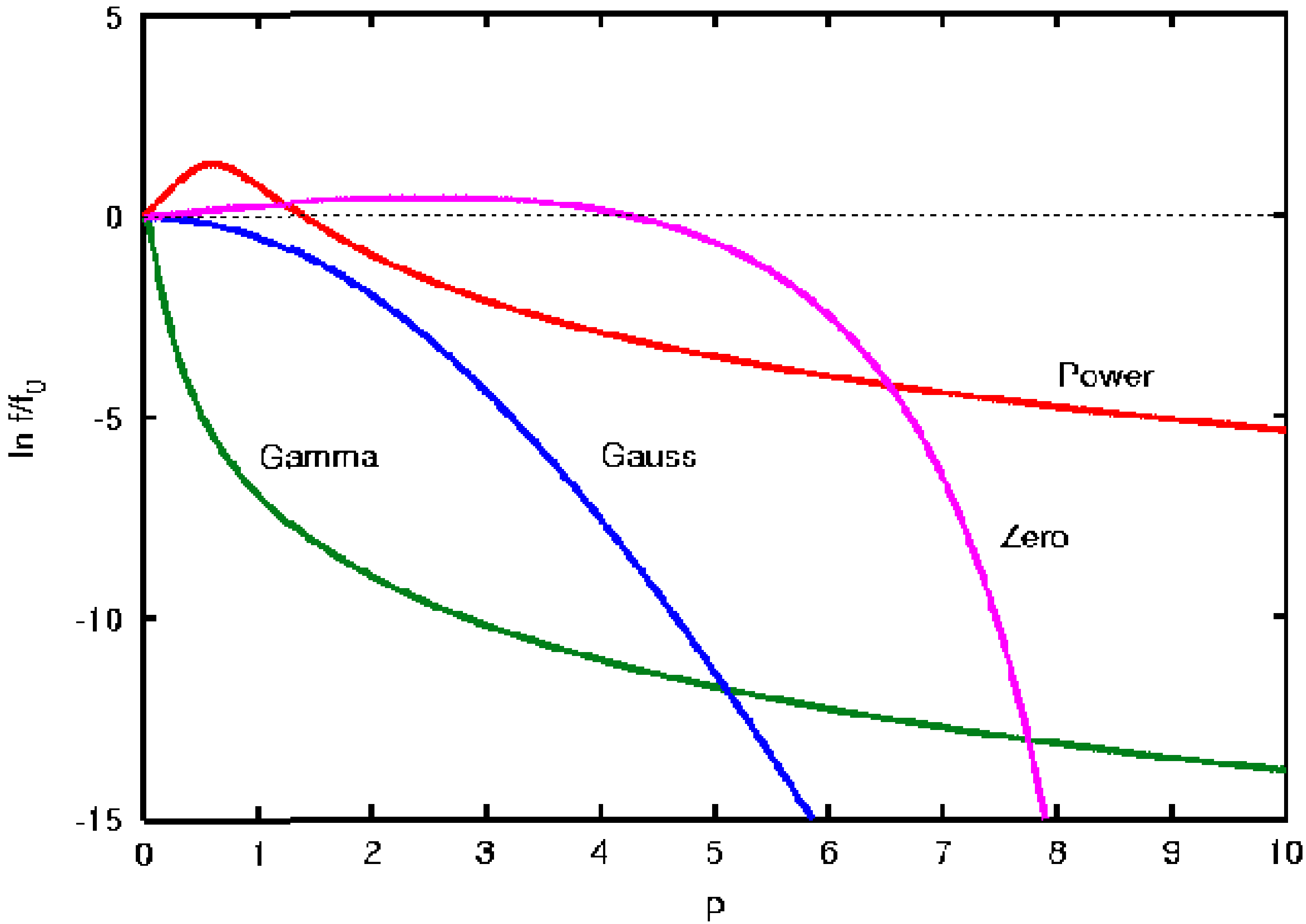
$E \gg E_c$: $f \sim (E/E_c)^{-v}$

Relation between slope, inflection and power !!

$$v = 1 + E_c / T$$

Stationary distributions

For F=0, B=0 the Tsallis distribution is the exact stationary solution



Gamma: p_c = 0.1 GeV, $F \neq 0$

Gauss: $p_c = \infty$

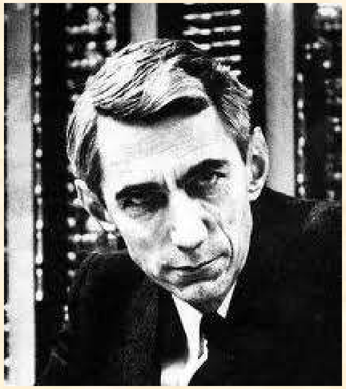
Zero: p_c = 10 GeV, $B^2 = D/C$

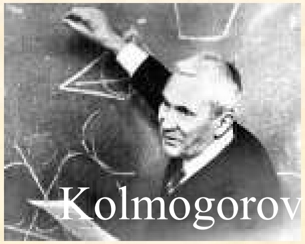
Power: p_c = 1 GeV, $F \neq 0$

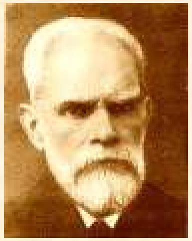
MODERN PROBLEMS

- Information
- Chaos
- Phase Transition
- Scaling
- Anomalous Fluctuations


Neumann


Shannon




Lyapunov


Rényi


Feynman


Wilson


Tsallis

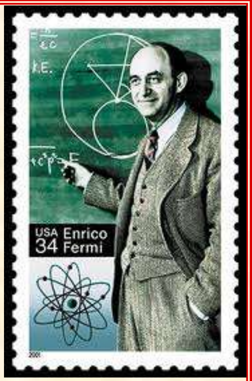

Lévy

Statistics, entropy, temperature

- Fermi distribution (Bernoulli, Poisson)
- Bose distribution (negative binomial)
- Superstatistics: distribution of distributions

Fermi distribution: N, K-N

$$W = \frac{K!}{N!(K-N)!} = \binom{K}{N}$$



$$\Omega = k_B \ln W - \beta\hbar\omega(N + 1/2) + \beta\mu N = \max$$

$$\Omega_{N+1} \le \Omega_N \quad \Omega_{N-1} \le \Omega_N$$

Fermi distribution: N, K-N

$$\text{Notation:}\quad \ln x = \beta(\hbar\omega - \mu), \quad k_B = 1.$$

$$\Omega_{N+1} - \Omega_N = \ln\frac{K-N}{N+1} - \ln x$$

$$\Omega_N - \Omega_{N-1} = \ln\frac{K-N+1}{N} - \ln x$$

$$\frac{K-N}{N+1} \le x \le \frac{K-N+1}{N}$$

Fermi distribution: N, K-N

$$\text{fix}: \quad \bar{f} = N/K, \quad K \to \infty$$

$$\frac{1-\bar{f}}{\bar{f}+1/K} \le x \le \frac{1-\bar{f}+1/K}{\bar{f}}$$

$$\frac{1}{1+x} - \frac{1}{K}\frac{x}{1+x} \le \bar{f} \le \frac{1}{1+x} + \frac{1}{K}\frac{1}{1+x}$$

Fermi distribution: N, K-N

$$\bar{f} \rightarrow w_{Fermi}(x) = \frac{1}{1+x}$$

$$w_{Fermi} = \frac{1}{e^{\beta(\hbar\omega - \mu)} + 1}$$

Fermi distribution in a subsystem

$$P_{n,k} = \frac{\begin{pmatrix} k \\ n \end{pmatrix}\begin{pmatrix} K-k \\ N-n \end{pmatrix}}{\begin{pmatrix} K \\ N \end{pmatrix}}$$

Fermi distribution in small subsystems

$$k << K, \quad \Rightarrow \quad n << N, \quad (N-n)! \approx N!\, N^{-n}$$

$$P_{n,k} \approx \frac{\begin{pmatrix} k \\ n \end{pmatrix}}{\begin{pmatrix} K \\ N \end{pmatrix}} \frac{K!\, K^{-k}}{N!\, N^{-n} (K-N)!\, (K-N)^{-(k-n)}}$$

Fermi distribution in small subsystems = Bernoulli distribution

$$P_{n,k} \approx \binom{k}{n} \left(\frac{K-N}{K} \right)^{k} \left(\frac{N}{K-N} \right)^{n}$$

$$P_{n,k} \approx \binom{k}{n} \bar{f}^{\,n} \, (1-\bar{f})^{k-n}$$

A story of false coins

Bose distribution in a sunsystem

$$P_{n,k} = \frac{\binom{k+n}{n}\binom{K-k+N-n}{N-n}}{\binom{K+N+1}{N}}$$

k levels and n excitations mixed arbitrarily…

Bose distribution in small subsystems

$$P_{n,k} \approx \binom{k+n}{n} \bar{f}^{\,n} \, (1+\bar{f})^{-k-1-n}$$

$$\langle n \rangle = (k+1)\bar{f}$$

$$\bar{f} = w_{\mathrm{Bose}} = \frac{1}{e^{\beta(\hbar\omega-\mu)}-1}$$

Negative binomial distribution (NBD)

$$\binom{k+n}{n} = (-1)^n \binom{-k-1}{n}$$

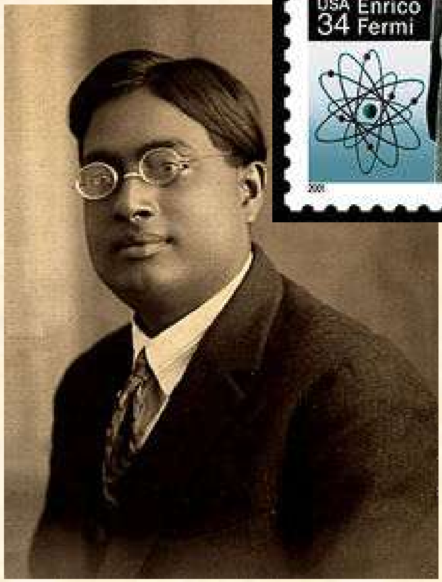
$$P_{n.k} \approx \binom{-k-1}{n} (-\bar{f})^n (1+\bar{f})^{-k-1-n}$$

Fermi – Bose transformation: statistical supersymmetry

$$B_{n,k}(\bar{f}) = F_{n,-k-1}(-\bar{f})$$

$$F_{n,k}(\bar{f}) = B_{n,-k-1}(-\bar{f})$$

invariant $\quad k(k+1)$



Rare events: Poisson distribution

$$P_n = \frac{a^n}{n!} e^{-a}$$

$$\langle n \rangle = a$$

$$\delta n^2 = a$$

Rare event Bernoulli: Poisson

$$n \ll k \quad \binom{k}{n} \approx \frac{k^n}{n!}$$

$$P_n \approx (1-\bar{f})^k \frac{1}{n!}\left(\frac{k\bar{f}}{1-\bar{f}}\right)^n$$

$$P_n = C_k(x)\frac{1}{n!}\left(ke^{-x}\right)^n$$

Rare event NBD: Poisson

$$n << k \quad \binom{k+n}{n} \approx \frac{k^n}{n!}$$

$$P_n \approx (1+\bar{f})^{-k-1} \frac{1}{n!} \left(\frac{k\bar{f}}{1+\bar{f}} \right)^n$$

$$P_n = C_k(x) \frac{1}{n!} \left(ke^{-x} \right)^n$$

CORRECTIONS

- **Finite Size Effects**
- **Near-Equilibrium Fluctuations**
- **Scaling Fluctuations**
- **Superstatistics**

NBD = Euler ○ Poisson

$$\int_0^\infty x^N e^{-ax}\, dx = \frac{N!}{a^{N+1}}$$

$$P_{n,k} = \binom{k+n}{n} \bar{f}^n (1+\bar{f})^{-k-1-n} =$$

$$\frac{\bar{f}^n}{k!\,n!} \int_0^\infty x^{k+n}\, e^{-(1+\bar{f})x}\, dx$$

NBD = Euler ○ Poisson

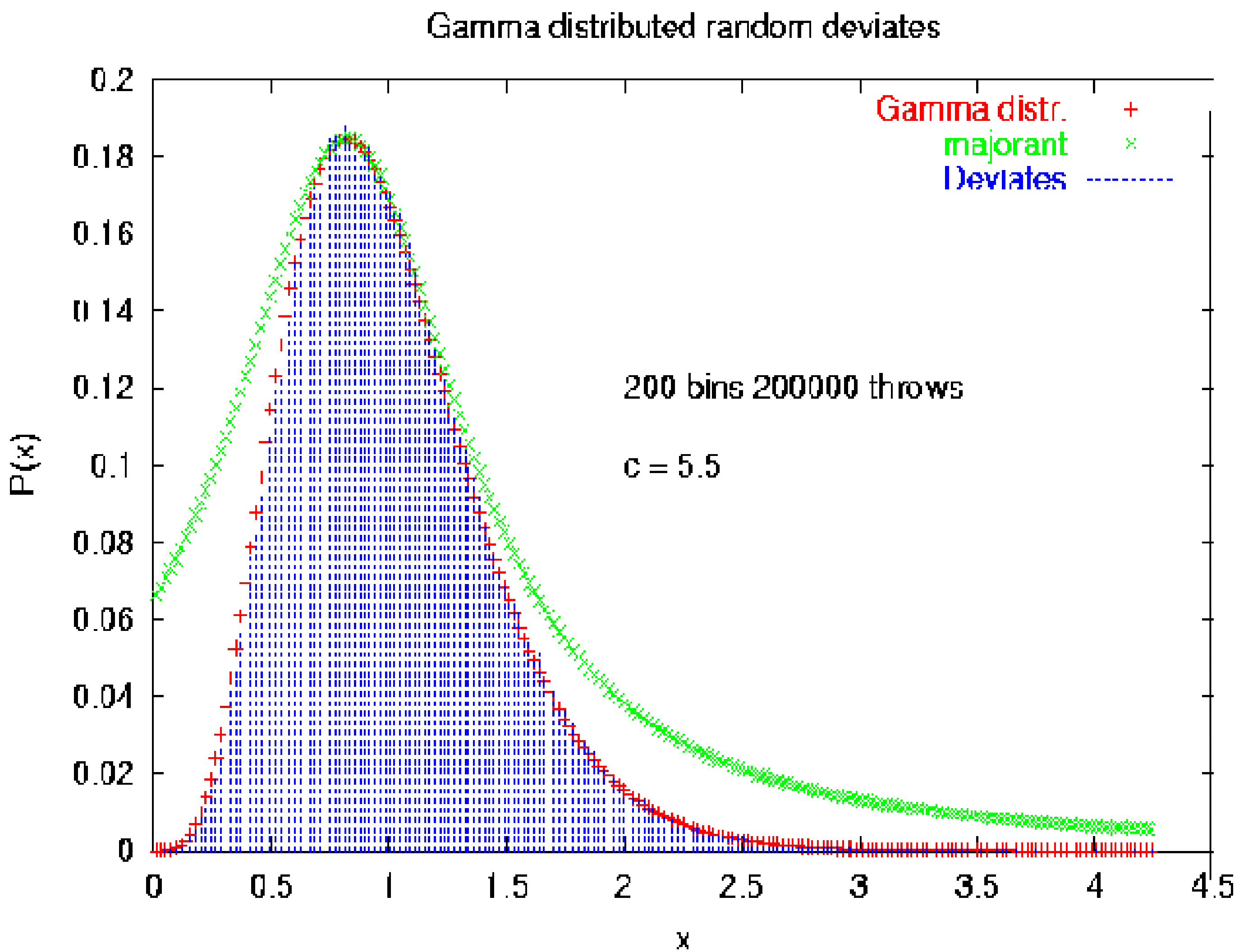
$$P_{n,k} = \int_0^{\infty} \frac{(x\bar{f})^n}{n!} e^{-\bar{f}x} \cdot \frac{x^k}{k!} e^{-x}\, dx$$

Poisson in k,   Euler-Gamma in x

Superstatistics

Euler - Gamma distribution

max: $1 - 1/c$, mean: 1, spread: $1/\sqrt{c}$



Homework problems

1. Regard the following distributions:
   - Bernoulli (n, k; f)
   - NBD (n, k; f)
   - Poisson (n, k; f)

Questions:
- Check the norm
- n expectation value, squared variance
- characteristic function (expectation value of exp(bn))

Homework problems

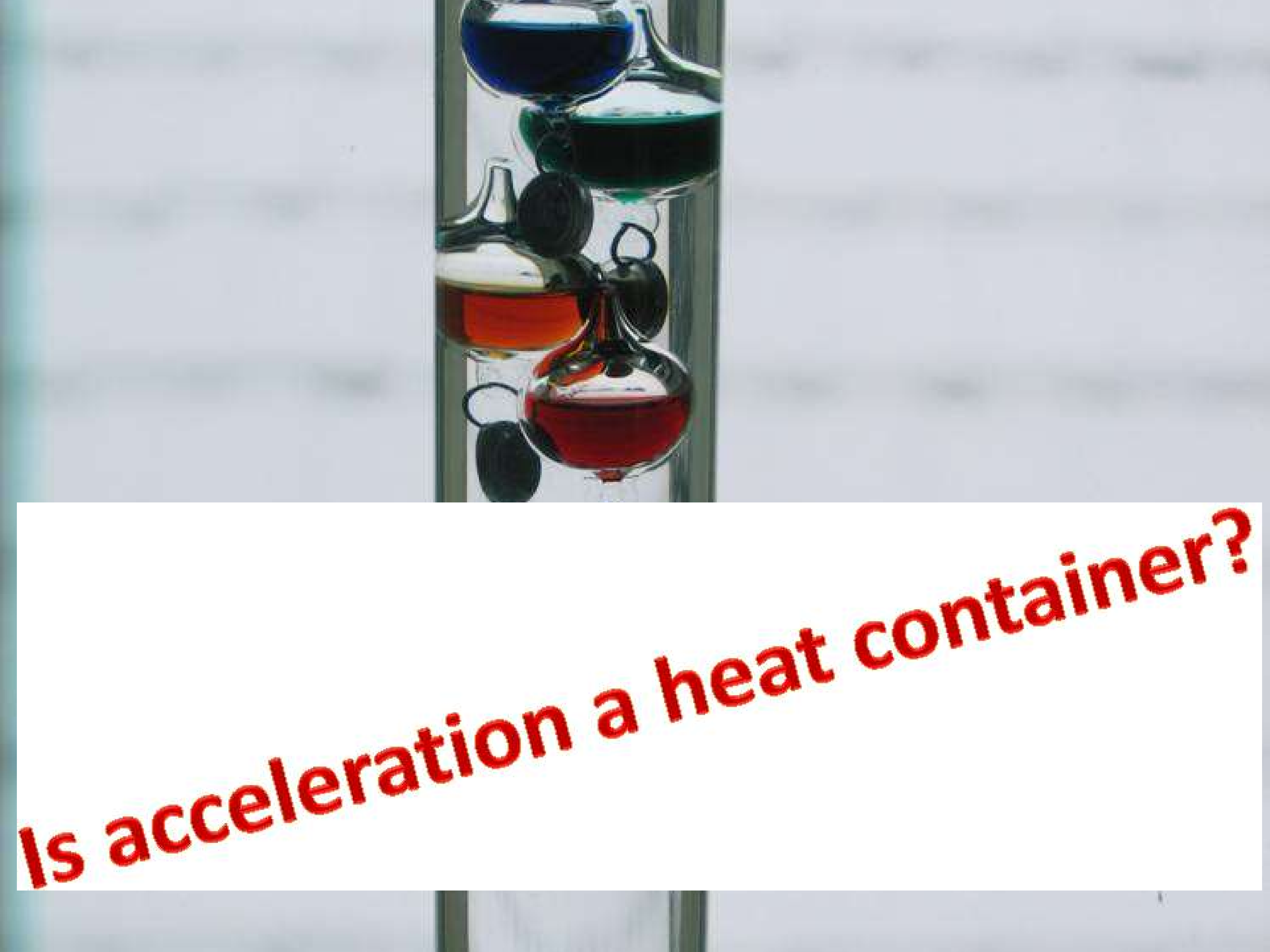
1. How is the supertransformation for finite subsystems inside finite systems ?

B(n; k | N; K) ←→ F(n; k | N; K)

What is the expectation value of f(a+x), if x is distributed as

a) Gauss

b) Euler-Gamma ?



Is acceleration a heat container?